# On a particular form of a symmetric Pöschl-Teller potential

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#### Abstract

We show that solutions of the Schrödinger equation with a symmetric Pöschl-Teller potential of a particular form can be expressed in terms of a closed combination (not series) of trigonometric functions. Using some properties of the eigenfunctions of the Schrödinger equation and their inner product we determine a new exact representation of the hypergeometric function with certain values of parameters in terms of a closed combination of trigonometric functions. We also obtain new results in an explicit closed form for integrals with the hypergeometric function and with the specific combination of trigonometric functions.

## 1 Solutions in terms of hypergeometric function

There are well known solutions of the Schrödinger equation

$$\left[ -\frac{d^2}{dx^2} + U(x) \right] \psi = E\psi \tag{1}$$

with the Pöschl-Teller potential

$$U(x) = \alpha^2 \left[ \frac{\kappa (\kappa - 1)}{\sin^2 (\alpha x)} + \frac{\lambda (\lambda - 1)}{\cos^2 (\alpha x)} \right]; \ \kappa > 1, \ \lambda > 1$$
 (2)

in the interval  $(0, \frac{\pi}{2\alpha})$ , which are expressed in terms of hypergeometric function  ${}_2F_1(\alpha, \beta; \gamma; x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \frac{x^n}{n!}$ :

$$\psi_n(x) = A_n \sin^{\kappa}(\alpha x) \cos^{\lambda}(\alpha x) {}_{2}F_1\left(-n, n+\kappa+\lambda; \kappa+\frac{1}{2}; \sin^{2}(\alpha x)\right), \ n \ge 0,$$
 (3)

 $A_n$  is a normalization factor, with energy spectrum

$$E_n = \alpha^2 \left(2n + \kappa + \lambda\right)^2 . \tag{4}$$

These solutions are described, for example, in Ref. [1], problem 38 and presented in Ref. [2], Appendix A, problem A.I.6. We consider a particular form of the Pöschl-Teller potential with values of the parameters  $\kappa = \lambda = 2$  and obtain solutions in terms of usual trigonometric functions in a closed form (not series).

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# 2 Solutions generated by Darboux transformation method

In order to obtain the solutions in terms of trigonometric functions we apply the Darboux transformation (DT) method [3], [4], [5] to the Schrödinger equation with the infinite rectangular well potential in the same interval  $(0, \frac{\pi}{2\alpha})$ . Then the initial operator of the DT method is the Hamiltonian

$$\widehat{H}_0 = -\frac{d^2}{dx^2}, \ x \in \left(0, \frac{\pi}{2\alpha}\right); \ \widehat{H}_0 \varphi_k = \varepsilon_k \varphi_k \ .$$
 (5)

Its eigenfunctions and energy spectrum are given by

$$\varphi_k(x) = \sqrt{\frac{4\alpha}{\pi}} \sin(2\alpha kx), \ \varepsilon_k = 4\alpha^2 k^2, \ k \ge 1.$$
(6)

We note that eigenfunctions  $\varphi_k$  in Eq. (6) are orthonormalized. In accordance with the DT method we represent  $\hat{H}_0$  in the form

$$\widehat{H}_0 = \widehat{L}^{\dagger} \widehat{L} + \omega^2 ,$$

where  $\omega^2$  is a real constant,  $\widehat{L}$  is the intertwining operator, and  $\widehat{L}^{\dagger}$  is its adjoint:

$$\widehat{L} = \frac{d}{dx} + W(x), \ \widehat{L}^{\dagger} = -\frac{d}{dx} + W(x). \tag{7}$$

Applying  $\widehat{L}$  to both sides of Eq. (5) one has

$$\widehat{L}\widehat{H}_0\varphi_k = \widehat{L}\left(\widehat{L}^{\dagger}\widehat{L} + \omega^2\right)\varphi_k = \left(\widehat{L}\widehat{L}^{\dagger} + \omega^2\right)\widehat{L}\varphi_k = \varepsilon_k\widehat{L}\varphi_k$$

that can be written as

$$\widehat{H}_1 \chi_k = \varepsilon_k \chi_k, \ \widehat{H}_1 = \widehat{L} \widehat{L}^\dagger + \omega^2, \ \chi_k = \widehat{L} \varphi_k \ ,$$
 (8)

where  $\chi_k$  are eigenfunctions of  $\widehat{H}_1$ . The spectrum  $\varepsilon_k$  of  $\widehat{H}_1$  is the same of  $\widehat{H}_0$ , except maybe some particular values. Using the relations (7), we write

$$\widehat{H}_{1} = -\frac{d^{2}}{dx^{2}} + V_{1}(x), \ V_{1}(x) = W' + W^{2} + \omega^{2}$$
(9)

Choosing W(x),  $\omega^2$  as follows

$$W(x) = -\frac{d}{dx}\ln(\varphi_1) = -\frac{\varphi'_1}{\varphi_1}, \ \omega^2 = \varepsilon_1 \ , \tag{10}$$

one has for them from Eq. (6)

$$W(x) = -2\alpha \cot(2\alpha x)$$
,  $\omega^2 = 4\alpha^2$ 

and for  $V_1(x)$  of Eq. (9)

$$V_1(x) = 2\left(\frac{\varphi_1'^2}{\varphi_1^2} + \varepsilon_1^2\right) = 2(2\alpha)^2 \left(\cot^2(2\alpha x) + 1\right) = \frac{8\alpha^2}{\sin^2(2\alpha x)}.$$
 (11)

To obtain the eigenfunctions of  $\widehat{H}_1$  by means of the DT method we apply  $\widehat{L}$  to the eigenfunctions  $\varphi_k$  of the initial Hamiltonian  $\widehat{H}_0$  that follows from Eq. (8)

$$\chi_k = \widehat{L}\varphi_k = \frac{d\varphi_k}{dx} + W(x)\varphi_k \tag{12}$$

$$= \sqrt{\frac{4\alpha}{\pi}} 2\alpha \left[ k \cos \left( 2\alpha k x \right) - \cot \left( 2\alpha x \right) \sin \left( 2\alpha k x \right) \right], \ k \ge 2.$$

The eigenfunctions of (12) are not normalized. To normalize them we use the following properties of an inner product of the eigenfunctions:

$$(\chi_i, \chi_j) = (\widehat{L}\varphi_i, \widehat{L}\varphi_j) = (\widehat{L}^{\dagger}\widehat{L}\varphi_i, \varphi_j) = ((\widehat{H}_0 - \omega^2)\varphi_i, \varphi_j)$$
$$= ((\varepsilon_i - \omega^2)\varphi_i, \varphi_j) = (\varepsilon_i - \omega^2)(\varphi_i, \varphi_j) = (\varepsilon_i - \omega^2)\delta_{ij}.$$

Designating the normalized eigenfunctions by  $\widetilde{\chi}_k = \widetilde{N}_k \chi_k$ , one has

$$(\widetilde{\chi}_k, \widetilde{\chi}_k) = (\widetilde{N}_k \chi_k, \widetilde{N}_k \chi_k) = \widetilde{N}_k^2 (\chi_k, \chi_k) = \widetilde{N}_k^2 (\varepsilon_k - \omega^2) = 1$$

therefore

$$\widetilde{N}_k = \frac{1}{\sqrt{\varepsilon_k - \omega^2}} \ .$$

For the choice of  $\omega^2$  indicated in Eq. (10) and with use of Eq. (6) one has

$$\widetilde{N}_k = \frac{1}{\sqrt{\varepsilon_k - \varepsilon_1}} = \frac{1}{2\alpha\sqrt{k^2 - 1}}$$
.

Then the normalized eigenfunctions and the energy spectrum are

$$\widetilde{\chi}_k(x) = N_k \left[ k \cos(2\alpha kx) - \cot(2\alpha x) \sin(2\alpha kx) \right], \ N_k = \sqrt{\frac{4\alpha}{\pi}} \frac{1}{\sqrt{k^2 - 1}}, \ \varepsilon_k = 4\alpha^2 k^2, \ k \ge 2 \ . \tag{13}$$

For the ground state energy one has

$$\varepsilon_2 = 4\alpha^2 2^2 \ . \tag{14}$$

We note that at the middle of the interval,  $x = \frac{\pi}{4\alpha}$ :

$$\widetilde{\chi}_k\left(\frac{\pi}{4\alpha}\right) = N_k\left[k\cos\left(k\frac{\pi}{2}\right)\right], \ \widetilde{\chi}'_k\left(\frac{\pi}{4\alpha}\right) = N_k2\alpha\left[1 - k^2\right]\sin\left(k\frac{\pi}{2}\right) \ ,$$
 (15)

then for even k = 2m:

$$\widetilde{\chi}_{2m}\left(\frac{\pi}{4\alpha}\right) = N_{2m} 2m \left(-1\right)^m, \ \widetilde{\chi}'_{2m}\left(\frac{\pi}{4\alpha}\right) = 0 \tag{16}$$

and for odd k = 2m + 1:

$$\widetilde{\chi}_{2m+1}\left(\frac{\pi}{4\alpha}\right) = 0, \ \widetilde{\chi}'_{2m+1}\left(\frac{\pi}{4\alpha}\right) = N_{2m+1}2\alpha \left[1 - (2m+1)^2\right] (-1)^m \ .$$
(17)

# 3 Correspondence between the solutions

The potential  $V_1(x)$  of Eq. (11) corresponds to the Pöschl-Teller potential U(x) of Eq. (2) with the values of the parameters  $\kappa = \lambda = 2$ :

$$U\left(x\right) = 2\alpha^{2} \left(\frac{1}{\sin^{2}\left(\alpha x\right)} + \frac{1}{\cos^{2}\left(\alpha x\right)}\right) = \frac{2\alpha^{2}}{\sin^{2}\left(\alpha x\right)\cos^{2}\left(\alpha x\right)} = \frac{8\alpha^{2}}{\sin^{2}\left(2\alpha x\right)}.$$

The eigenfunctions and the energy in the form of Eqs. (3), (4) with  $\kappa = \lambda = 2$  are

$$\psi_n(x) = A_n \sin^2(\alpha x) \cos^2(\alpha x) \,_2F_1\left(-n, n+4; \frac{5}{2}; \sin^2(\alpha x)\right) , \qquad (18)$$

$$E_n = \alpha^2 (2n+4)^2 = 4\alpha^2 (n+2)^2 , \ n \ge 0 .$$

For the ground state energy one has

$$E_0 = 4\alpha^2 2^2 \ . \tag{19}$$

Taking into account Eqs. (14), (19) and Eqs. (13), (18), the following correspondence takes place for energies and the normalized eigenfunctions:

$$E_n = \varepsilon_{n+2}, \ \psi_n(x) = \widetilde{\chi}_{n+2}(x), \ n \ge 0.$$
 (20)

where due to Eq. (13)  $\widetilde{\chi}_{n+2}(x)$  is

$$\widetilde{\chi}_{n+2}(x) = N_{n+2} \left[ (n+2)\cos(2(n+2)\alpha x) - \cot(2\alpha x)\sin(2(n+2)\alpha x) \right],$$

$$N_{n+2} = \sqrt{\frac{4\alpha}{\pi}} \frac{1}{\sqrt{(n+2)^2 - 1}}, \ n \ge 0.$$
(21)

We also note that at the middle of the interval,  $x = \frac{\pi}{4\alpha}$ :

$$\psi_n\left(\frac{\pi}{4\alpha}\right) = A_n \frac{1}{4} \,_2F_1\left(-n, n+4; \frac{5}{2}; \frac{1}{2}\right) \,\,\,\,\,(22)$$

$$\psi_n'\left(\frac{\pi}{4\alpha}\right) = A_n \frac{\alpha}{10} \left(-n\right) \left(n+4\right) {}_{2}F_1\left(-n+1, n+5; \frac{7}{2}; \frac{1}{2}\right) , \qquad (23)$$

where the relation

$$\frac{d}{dx} {}_{2}F_{1}\left(\alpha,\beta;\gamma;x\right) = \frac{\alpha\beta}{\gamma} {}_{2}F_{1}\left(\alpha+1,\beta+1;\gamma+1;x\right)$$

was used.

# 4 Some new properties of the hypergeometric function

Now we are at position to establish an expression of the hypergeometric function of the form of Eq. (18) in terms of a composition of trigonometric functions. We write the normalization factor  $A_n$  of

Eq. (18) as follows

$$A_n = C_n^{-1} N_{n+2}, \ N_{n+2} = \sqrt{\frac{4\alpha}{\pi}} \frac{1}{\sqrt{(n+2)^2 - 1}} \ .$$
 (24)

Then from Eq. (20) and Eqs. (18), (21) one has

$${}_{2}F_{1}\left(-n,n+4;\frac{5}{2};\sin^{2}\left(\alpha x\right)\right) = 4C_{n}\frac{\left[\left(n+2\right)\cos\left(2\alpha\left(n+2\right)x\right) - \cot\left(2\alpha x\right)\sin\left(2\alpha\left(n+2\right)x\right)\right]}{\sin^{2}\left(2\alpha x\right)}.$$
(25)

To determine  $C_n$  in Eq. (25) we write Eq. (20) for the eigenfunctions and their derivatives at the point  $x = \frac{\pi}{4\alpha}$ . With use of Eq. (15) one has

$$\widetilde{\chi}_{n+2}\left(\frac{\pi}{4\alpha}\right) = -N_{n+2}\left[\left(n+2\right)\cos\left(n\frac{\pi}{2}\right)\right], \ \widetilde{\chi}'_{n+2}\left(\frac{\pi}{4\alpha}\right) = N_{n+2}2\alpha\left[\left(n+2\right)^2 - 1\right]\sin\left(n\frac{\pi}{2}\right) \ . \tag{26}$$

Then from Eqs. (22), (23), (26) for even n = 2m:

$$\frac{1}{4} {}_{2}F_{1}\left(-2m, 2m+4; \frac{5}{2}; \frac{1}{2}\right) = C_{2m}2(m+1)(-1)^{m+1} , \qquad (27)$$

therefore

$$C_{2m} = \frac{(-1)^{m+1}}{8(m+1)} {}_{2}F_{1}\left(-2m, 2m+4; \frac{5}{2}; \frac{1}{2}\right) ;$$
 (28)

for odd n = 2m + 1:

$$\frac{\alpha}{10} \left( -(2m+1)\right) (2m+5) {}_{2}F_{1} \left( -2m, 2m+6; \frac{7}{2}; \frac{1}{2} \right) = C_{2m+1} 2\alpha \left[ (2m+3)^{2} - 1 \right] (-1)^{m} , \quad (29)$$

therefore

$$C_{2m+1} = \frac{(-1)^{m+1}}{20} \frac{(2m+1)(2m+5)}{4(m+1)(m+2)} {}_{2}F_{1}\left(-2m, 2m+6; \frac{7}{2}; \frac{1}{2}\right) . \tag{30}$$

We also note that due to the properties of the solutions (16), (17) one can conclude that

$$_{2}F_{1}\left(-\left(2m+1\right),2m+5;\frac{5}{2};\frac{1}{2}\right)=0,\ _{2}F_{1}\left(-2m+1,2m+5;\frac{7}{2};\frac{1}{2}\right)=0,\ m=0,1,2,\dots$$
 (31)

Substituting Eqs. (28), (30) into Eq. (25), one obtains explicitly

$$\frac{{}_{2}F_{1}\left(-2m,2m+4;\frac{5}{2};\sin^{2}\left(\alpha x\right)\right)}{{}_{2}F_{1}\left(-2m,2m+4;\frac{5}{2};\frac{1}{2}\right)} = \frac{\left(-1\right)^{m+1}}{2\left(m+1\right)} \times \frac{\left[\left(2m+2\right)\cos\left(2\alpha\left(2m+2\right)x\right) - \cot\left(2\alpha x\right)\sin\left(2\alpha\left(2m+2\right)x\right)\right]}{\sin^{2}\left(2\alpha x\right)} \tag{32}$$

and

$$\frac{{}_{2}F_{1}\left(-\left(2m+1\right),2m+5;\frac{5}{2};\sin^{2}\left(\alpha x\right)\right)}{{}_{2}F_{1}\left(-2m,2m+6;\frac{7}{2};\frac{1}{2}\right)} = \frac{\left(-1\right)^{m+1}}{20}\frac{\left(2m+1\right)\left(2m+5\right)}{\left(m+1\right)\left(m+2\right)} \times \frac{\left[\left(2m+3\right)\cos\left(2\alpha\left(2m+3\right)x\right) - \cot\left(2\alpha x\right)\sin\left(2\alpha\left(2m+3\right)x\right)\right]}{\sin^{2}\left(2\alpha x\right)}.$$
(33)

Besides that we present results of some integrals with the combination of trigonometric functions of Eq. (13) and with the hypergeometric function of Eq. (18).

1. As far as the eigenfunctions  $\widetilde{\chi}_k$  are normalized,  $\int_0^{\frac{\pi}{2\alpha}} \widetilde{\chi}_k^2 dx = 1$ , from Eq. (13) one has

$$\int_0^{\frac{\pi}{2\alpha}} \left[ k \cos(2\alpha kx) - \cot(2\alpha x) \sin(2\alpha kx) \right]^2 dx = \frac{\pi}{4\alpha} \left( k^2 - 1 \right) .$$

Passing to integration over the inerval  $(0,\pi)$  by a change of variable  $2\alpha x \to x$ , one gets

$$\int_0^{\pi} \left[ k \cos(kx) - \cot(x) \sin(kx) \right]^2 dx = \frac{\pi}{2} \left( k^2 - 1 \right) . \tag{34}$$

Writing the eigenfunctions in terms of the hypergeometric function, Eqs. (18), (24), one has

$$\int_{0}^{\frac{\pi}{2\alpha}} \sin^{4}(\alpha x) \cos^{4}(\alpha x) \,_{2}F_{1}^{2}\left(-n, n+4; \frac{5}{2}; \sin^{2}(\alpha x)\right) dx = \frac{\pi}{4\alpha} \left((n+2)^{2}-1\right) C_{n}^{2}.$$

Passing to integration over the inerval  $(0, \pi/2)$  by a change of variable  $\alpha x \to x$ , one gets

$$\int_0^{\frac{\pi}{2}} \sin^4 x \cos^4 x \, {}_2F_1^2 \left( -n, n+4; \frac{5}{2}; \sin^2(x) \right) dx = \frac{\pi}{4} \left( (n+2)^2 - 1 \right) C_n^2 \,. \tag{35}$$

Making a change of variable  $z = \sin^2 x$  in Eq. (35), one obtains

$$\int_0^1 z^{\frac{3}{2}} (1-z)^{\frac{3}{2}} {}_2F_1^2 \left(-n, n+4; \frac{5}{2}; z\right) dz = \frac{\pi}{2} \left((n+2)^2 - 1\right) C_n^2.$$
 (36)

The coefficient  $C_n$  in Eqs. (35) and (36) must be taken in dependence of even or odd n as

$$C_n^{even} = \frac{(-1)^{\frac{n}{2}+1} {}_{2}F_{1}\left(-n, n+4; \frac{5}{2}; \frac{1}{2}\right)}{4(n+2)},$$

$$C_n^{odd} = \frac{(-1)^{\frac{n-1}{2}+1} n(n+4) {}_{2}F_{1}\left(-n+1, n+5; \frac{7}{2}; \frac{1}{2}\right)}{20\left((n+2)^{2}-1\right)}.$$
(37)

2. One can demostrate that  $\widetilde{\chi}_k^2(x)$  is symmetric relative to the middle of the interval  $x = \frac{\pi}{4\alpha}$ :

$$\widetilde{\chi}_k^2 \left( -\left(x - \frac{\pi}{4\alpha}\right) \right) = \widetilde{\chi}_k^2 \left(x - \frac{\pi}{4\alpha}\right) .$$

Then one can write for the expected value of the coordinate x in a state  $\tilde{\chi}_n$ :

$$\langle x \rangle_{k} = \int_{0}^{\frac{\pi}{2\alpha}} \widetilde{\chi}_{k}^{*} \widehat{x} \widetilde{\chi}_{k} dx = \int_{0}^{\frac{\pi}{2\alpha}} \widetilde{\chi}_{k}^{2}(x) x dx = \int_{-\frac{\pi}{4\alpha}}^{\frac{\pi}{4\alpha}} \widetilde{\chi}_{k}^{2}(\xi) \left(\xi + \frac{\pi}{4\alpha}\right) d\xi$$

$$= \frac{\pi}{4\alpha} \int_{-\frac{\pi}{4\alpha}}^{\frac{\pi}{4\alpha}} \widetilde{\chi}_{k}^{2}(\xi) d\xi = \frac{\pi}{4\alpha} ,$$
(38)

where  $\xi = x - \frac{\pi}{4\alpha}$ . Substituting  $\widetilde{\chi}_k(x)$  from Eq. (13) into Eq. (38), one has

$$\int_0^{\frac{\pi}{2\alpha}} \left[ k \cos\left(2\alpha k x\right) - \cot\left(2\alpha x\right) \sin\left(2\alpha k x\right) \right]^2 x dx = \frac{\pi^2}{16\alpha^2} \left( k^2 - 1 \right) .$$

Passing to integration over the inerval  $(0,\pi)$  by a change of variable  $2\alpha x \to x$ , one gets

$$\int_0^{\pi} \left[ k \cos(kx) - \cot(x) \sin(kx) \right]^2 x dx = \frac{\pi^2}{4} \left( k^2 - 1 \right) . \tag{39}$$

For the eigenfunctions in terms of the hypergeometric function, Eqs. (18), (24), one has

$$\int_{0}^{\frac{\pi}{2\alpha}} \sin^{4}(\alpha x) \cos^{4}(\alpha x) \,_{2}F_{1}^{2}\left(-n, n+4; \frac{5}{2}; \sin^{2}(\alpha x)\right) x dx = \frac{\pi^{2}}{16\alpha^{2}} \left((n+2)^{2}-1\right) C_{n}^{2}$$

Passing to integration over the inerval  $(0, \pi/2)$  by a change of variable  $\alpha x \to x$ , one gets

$$\int_0^{\frac{\pi}{2}} \sin^4(x) \cos^4(x) \,_2F_1^2\left(-n, n+4; \frac{5}{2}; \sin^2(x)\right) x dx = \frac{\pi^2}{16} \left((n+2)^2 - 1\right) C_n^2 \,. \tag{40}$$

The coefficient  $C_n$  in Eq. (40) must be taken in dependence of even or odd n as is given in Eq. (37).

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